§26 Integration and Differentiation of Power Series

Consider a power series $\sum_{n=0}^{\infty} a_n x^n$. If you differentiate term-by-term, you would get the series $\sum_{n=0}^{\infty} n a_n x^{n-1}$; if you integrate term-by-term, you would get $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$. Our goal here is to establish when these operations can be performed, or more specifically we wish to show that if $f(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$ then $f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$ and $F(x) = \int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$. The basic message is that these operations are valid if the original series has radius of convergence $R > 0$ and if $|x| < R$.

First, we prove a theorem that says that if you integrate or differentiate a series term-by-term, the radius of convergence is unchanged.

Theorem. Suppose $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then so do $\sum_{n=0}^{\infty} n a_n x^{n-1}$ and $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$.

Proof. When we proved the theorem that power series have a radius of convergence R, we showed that the radius of convergence is $R = 1/\beta$, where $\beta = \limsup_{n\to\infty} |a_n|^{1/n}$.

Now the series $\sum_{n=0}^{\infty} n a_n x^{n-1}$ converges exactly when the series $\sum_{n=0}^{\infty} n a_n x^n$ converges. If we figure out β for this series, we would have $\beta = \limsup_{n\to\infty} |n a_n|^{1/n} = \limsup_{n\to\infty} n^{1/n} |a_n|^{1/n}$. Now $\lim_{n\to\infty} n^{1/n} = 1$ so we have $\beta = \limsup_{n\to\infty} |a_n|^{1/n}$. (There is a theorem, 12.1 in Ross, that says that $\limsup(s_n t_n) = s \limsup t_n$ if $\lim s_n = s$.) So β is unchanged by differentiating the series, so the radius of convergence does not change either.

The series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ converges exactly when the series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^n$ converges. This time, we have $\beta = \limsup_{n\to\infty} \left|\frac{a_n}{n+1}\right|^{1/n} = \limsup_{n\to\infty} \left|\frac{1}{n+1}\right|^{1/n} |a_n|^{1/n}$. This time, $\lim_{n\to\infty} \left|\frac{1}{n+1}\right|^{1/n} = 1$, so again β is the same, and so is the radius of convergence.

$\square$

This theorem by itself doesn't guarantee that if f is represented by a series, then f' is represented by the derivative of the series, or that F is represented by the integral of the series. First, we consider integration. The key theorem is the following:

Theorem. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. Suppose $0 < c < R$. Then the power series converges uniformly to a continuous function on $[-c, c]$.

Proof. This is an application of the Weierstrass M-test. That test says that a series of functions $\sum_{n=0}^{\infty} g_n$ converges uniformly on a set S if there are numbers $M_n > 0$ for $n \geq 0$ such that $\sum_{n=0}^{\infty} M_n < \infty$ and if $|g_n(x)| \leq M_n$ for all $x \in S$ and $n \geq 0$. Here, we let $S = [-c, c]$, we let $g_n(x) = a_n x^n$ and we let $M_n = |a_n c^n|$. Since $0 < c < R$, we are assured that $\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} |a_n| c^n$ converges. We only need to verify that if $x \in [-c, c]$ that $|g_n(x)| = |a_n x^n| \leq |a_n| c^n$. But this is clear since x^n is an increasing function on $[0, c]$ (so $|x|^n \leq |c|^n$ if $|x| \leq |c|$). So the convergence of the power series is uniform. Since the partial sums are polynomials and hence continuous, we know that their uniform limit must be a continuous function. □

Since the power series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-c, c]$, for $0 < c < R$, we can integrate it by the theorem that says if $f_n \to f$ uniformly on $[a, b]$ then $\lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b f(t)\, dt$. We have the following theorem:

Theorem. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence $R > 0$. Let $F(x) = \int_0^x f(t)\, dt$. Then $F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ for all $|x| < R$.

Proof. First, consider the case $-R < x \leq 0$. We know from the theorem above that the series $f(t) = \sum_{n=0}^{\infty} a_n t^n$ converges uniformly on the set $[x, 0]$. That is, the sequence $f_n(t) = \sum_{k=0}^{n} a_k t^k$ of partial sums converges uniformly to the continuous function $f(t)$ on the interval $[x, 0]$. So we can integrate:

$$\int_x^0 f_n(t)\,dt = \int_x^0 \left(\sum_{k=0}^{n} a_k t^k\right) dt = \left[\sum_{k=0}^{n} \frac{a_k}{k+1} t^{k+1}\right]_x^0 = -\sum_{k=0}^{n} \frac{a_k}{k+1} x^{k+1}.$$

Therefore,

$$\begin{aligned} F(x) = \int_0^x f(t)\,dt = -\int_x^0 f(t)\,dt &= -\lim_{n\to\infty} \int_x^0 f_n(t)\,dt \\ &= -\lim_{n\to\infty}\left(-\sum_{k=0}^{n} \frac{a_k}{k+1} x^{k+1}\right) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}. \end{aligned}$$

□

Now the last theorem of this section: we can differentiate a power series term-by-term inside its interval of convergence.

Theorem. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence $R > 0$. Then for $|x| < R$ we have $f'(t) = \sum_{n=0}^{\infty} n a_n x^{n-1}$.

Proof. There is no theorem that says that if a sequence of functions converges uniformly, then their derivatives converge. (It isn't hard to construct an example where this is false: let $f_n(x) = \frac{\sin nx}{n}$. It's easy to show $f_n \to 0$ uniformly but $f_n'(x) = \cos nx$ doesn't converge to 0.)

But this theorem is easy to prove using the previous theorem. Let $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$. We recall that g has the same radius of convergence as does f. So we can integrate this series term-by-term. For $|x| < R$ we get $G(x) = \int_0^x g(t)\,dt = \sum_{n=1}^{\infty} a_n x^n = f(x) - a_0$. By the Fundamental Theorem of Calculus, we have $G'(x) = g(x) = f'(x)$. So we are done.

□

In our notes for section 23, we demonstrated the power of these theorems by developing a power series for the inverse tangent function, by integrating the geometric series for $\frac{1}{1+t^2}$. Here is another example:

Example. Consider the series, valid for $|t| < 1$,

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + t^4 + \cdots .$$

Integrate this for $|x| < 1$ to obtain a series for the natural logarithm function:

$$-\ln|1-x| = \int_0^x \frac{1}{1-t}\,dt = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots .$$

This series is actually valid not just for $-1 < x < 1$ but for $x = -1$ as well, which produces the series

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots .$$

This follows from Abel's Theorem, which is stated and proved in Ross. We omit that theorem since it is a bit tricky. The same theorem also proves the series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

Example. Define the function $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. It isn't hard (using the ratio test) to verify that this series has radius of convergence $R = \infty$ (that is, it converges for all x). So we are allowed to differentiate term-by-term. But it turns out if you differentiate this series, you get the exact same series. That is, $E'(x) = E(x)$.

Now suppose F is another function with this property, so $F'(x) = F(x)$. Then

$$\frac{d}{dx}\left(\frac{F(x)}{E(x)}\right) = \frac{F'(x)E(x) - F(x)E'(x)}{\big(E(x)\big)^2} = \frac{F(x)E(x) - F(x)E(x)}{\big(E(x)\big)^2} = 0.$$

This forces $\frac{F(x)}{E(x)} = C$ for all x, for some constant C. That is, there is a constant C such that $F(x) = CE(x)$ for all x.

Now let a be any number and define $F(x) = E(x+a)$. Then $F'(x) = E'(x+a) = E(x+a) = F(x)$. Therefore there is a constant C such that $F(x) = CE(x)$ for all x, that is, $E(x+a) = CE(x)$ for all x. Now let $x = 0$. We get $E(a) = CE(0)$ but $E(0) = 1$ so $E(a) = C$. This shows that for all x and a that $E(x+a) = E(a)E(x)$. Put differently, for all a and b, we have $E(a+b) = E(a)E(b)$. This is the salient algebraic property of the exponential function: $\exp(a+b) = \exp(a)\exp(b)$, that is, $e^{a+b} = e^a e^b$. You may recall from calculus that in fact, $E(x) = e^x$.

The point here is that by using some elementary calculus (and our theory on power series), we are able to define the natural exponential function as a power series and prove its properties – from scratch. See §37 in Ross for more information about defining elementary functions and developing their properties, using power series (and other techniques).