§25 Uniform Convergence of Sequences of Functions

We begin with the following Theorem:

Theorem. Suppose (f_n) is a sequence of continuous functions that converges uniformly to the function f on the interval $[a,b]$. Then

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx=\int_a^b f(x)\,dx.$$

Proof. Note if $f_n \to f$ uniformly and f_n is continuous for all n, then by a theorem from section 24 we have that f is continuous. We assume that we can integrate any continuous function, so we needn't worry about the existence of the integral of the limit function.

We need the following inequality: Suppose h is continuous on $[a,b]$. Then

$$\left|\int_a^b h(x)\,dx\right|\le\int_a^b |h(x)|\,dx.$$

To prove this inequality, note that for all $x\in[a,b]$,

$$-|h(x)|\le h(x)\le|h(x)|$$

so we integrate this inequality to obtain

$$-\int_a^b |h(x)|\,dx\le\int_a^b h(x)\,dx\le\int_a^b |h(x)|\,dx$$

which implies the inequality we are trying to prove (since $|u|<a\iff -a<u<a$).

Now let $\epsilon>0$. We would like to show the following: There is N such that $n>N$ implies

$$\left|\int_a^b f_n(x)\,dx-\int_a^b f(x)\,dx\right|<\epsilon.$$

Since $f_n\to f$ uniformly, we know there is N such that $n>N$ implies that $|f_n(x)-f(x)|<\dfrac{\epsilon}{b-a}$ for all $x\in[a,b]$. Therefore, if $n>N$ we have

$$\left|\int_a^b f_n(x)\,dx-\int_a^b f(x)\,dx\right|\le\int_a^b|f_n(x)-f(x)|\,dx<\int_a^b\frac{\epsilon}{b-a}\,dx=\epsilon.$$

So we are done. □

Soon, we will apply this theorem to the integration of power series. But we need another theorem or two about convergence of sequences of functions.

First, recall the definition of a sequence of (ordinary) numbers being Cauchy: (s_n) is Cauchy if for every $\epsilon > 0$ there is N such that $m, n > N$ implies $|s_m - s_n| < \epsilon$. Also, recall the point of this: if a sequence is Cauchy, it converges to something. So there is some s such that $\lim s_n = s$, if (s_n) is Cauchy. We want a similar theorem for sequences of functions. So we define a sequence of functions being "uniformly Cauchy":

Definition. Suppose (f_n) is a sequence of functions defined on a set $S \subset \mathbb{R}$. Then we say (f_n) is **uniformly Cauchy** if for every $\epsilon > 0$ we have N such that $|f_n(x) - f_m(x)| < \epsilon$ for every $x \in S$ and for every $m, n > N$.

Theorem. Suppose (f_n) is a sequence of functions that is uniformly Cauchy on a set S. Then there is a function f on S such that $f_n \to f$ uniformly.

Proof. First, we have to find f. Let x_0 be any number in S. Then consider the sequence $(f_n(x_0))$. This is a sequence of numbers (not functions). It happens to be a Cauchy sequence. This is because (f_n) is uniformly Cauchy: For every $\epsilon > 0$ there is N such that $|f_n(x) - f_m(x)| < \epsilon$ for every $x \in S$. So for every $\epsilon > 0$ there is N such that $|f_n(x_0) - f_m(x_0)| < \epsilon$, since $x_0 \in S$. This is exactly the definition that the sequence $(f_n(x_0))$ is Cauchy. Now since $(f_n(x_0))$ is Cauchy, it converges to something. Call its limit $f(x_0)$. So for each $x_0 \in S$, there is a number $f(x_0)$ that $(f_n(x_0))$ converges to. This defines the function f.

At this point, we know that the sequence of functions (f_n) converges point-wise to f on S. We need to show (f_n) converges uniformly to this function.

Let $\epsilon > 0$. We seek N such that $n > N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in S$. Since (f_n) is uniformly Cauchy, we know that there is N such that $m, n > N$ implies $|f_n(x) - f_m(x)| < \epsilon/2$ for all $x \in S$. This N will work.

Pick any $x \in S$. Then we know for that x that $(f_n(x))$ converges to $f(x)$, so there is some number $m > N$ such that $|f_m(x) - f(x)| < \epsilon/2$. Now let $n > N$. It follows for our $x \in S$ that

$$\begin{aligned} |f_n(x) - f(x)| &= |f_n(x) - f_m(x) + f_m(x) - f(x)| \\ &\le |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

So for any $x \in S$ we have shown that if $n > N$ then $|f_n(x) - f(x)| < \epsilon$. Here, N doesn't depend on x, so what we have shown is that $|f_n(x) - f(x)| < \epsilon$ whenever $x \in S$ and $n > N$. This is what we needed to show, so we are done. $\square$

Theorem. Suppose (f_n) is a sequence of continuous functions on a set S, and suppose (f_n) is uniformly Cauchy. Then there is a continuous function f on S such that $f_n \to f$

uniformly.

Proof. From the previous theorem, we know that there is a function f such that $f_n \to f$ uniformly. From an earlier theorem (in section 24), we know that if $f_n \to f$ uniformly and each f_n is continuous, so is f. So we are done. $\square$

Our goal is to consider how power series converge. We will apply this theorem to the partial sums of a power series, which are just polynomials and therefore continuous. We will apply this theorem in the following form: the Weierstrass M-test:

Theorem. Suppose we have a sequence of functions (g_n), each defined on a set S, such that $|g_n(x)| \leq M_n$ for all $x \in S$, for some number $M_n > 0$. Suppose further that g_n is continuous on S for all n and that the series $\sum_{n=1}^{\infty} M_n$ is convergent. Then the series of functions $\sum_{n=1}^{\infty} g_n$ converges uniformly to some continuous function g on the set S.

Proof. What we do here is show that the sequence of partial sums, $\left(\sum_{k=1}^{n} g_k\right)$ is uniformly Cauchy; since each (finite) partial sum of the series is a continuous function, the result would follow immediately from the last theorem. So let $\epsilon > 0$. What we need to show is that there is an N such that $m, n > N$ implies

$$\left|\sum_{k=1}^{n} g_k(x) - \sum_{k=1}^{m} g_k(x)\right| < \epsilon$$

for all $x \in S$.

Now the series $\sum_{n=1}^{\infty} M_n$ is convergent (and each term is non-negative), so there is an N such that if $m, n > N$ then $\left|\sum_{k=m}^{n} M_k\right| < \epsilon$. (This really says that the partial sums of the series form a Cauchy sequence, since the series converges; see Ross §14 or my notes for the same section.)

Therefore, if $x \in S$ and $n > m > N$, we have

$$\begin{aligned}\left|\sum_{k=1}^{n} g_k(x) - \sum_{k=1}^{m} g_k(x)\right| &= \left|\sum_{k=m+1}^{n} g_k(x)\right| \\ &\leq \sum_{k=m+1}^{n} |g_k(x)| \leq \sum_{k=m+1}^{n} M_k = \left|\sum_{k=m+1}^{n} M_k\right| < \epsilon.\end{aligned}$$

$\square$

Example. The power series $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$ converges uniformly on the interval $[-2, 2]$. We use the Weierstrass M-test to show this. For $-2 \le x \le 2$, we have

$$\left| \frac{x^n}{3^n} \right| \le \frac{2^n}{3^n} = \left(\frac{2}{3} \right)^n .$$

So if we let $g_n(x) = x^n/3^n$ and $M_n = \left(\frac{2}{3}\right)^n$, we know that the power series converges uniformly, since we have that $|g_n(x)| \le M_n$ and that $\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1 - \frac{2}{3}} = 3.$

Note, the radius of convergence of this series is $R = 3$, so we know that it converges (pointwise) on the interval $(-3, 3)$. (This is its exact interval of convergence.) We have only shown it is uniformly convergent on $[-2, 2]$. Actually, the same argument can be used to show that it is uniformly convergent on any interval $[-c, c]$ where $0 < c < 3$. But it is *not* uniformly convergent on $(-3, 3)$. [See Ross, examples 4 and 5 of §25.]

The point of this example is this: By verifying that the power series converges uniformly, we are then justified in integrating it term-by-term, by the first theorem in these notes. We will so this sort of thing in the next section.