Test 2 Solutions

1. Use the Weierstrass M-Test to show that the series $g(x) = \sum_{n=1}^{\infty} \frac{1}{3^n} \cos\left(\frac{x}{2^n}\right)$ converges uniformly on $\mathbb{R}$.

We have $\left|\frac{1}{3^n} \cos\left(\frac{x}{2^n}\right)\right| \leq \frac{1}{3^n}$ so we let $M_n = \frac{1}{3^n}$. Now $\sum_{n=1}^{\infty} M_n = \frac{1}{2}$ so by the M-Test the series converges uniformly on $\mathbb{R}$.

2. Prove the following Theorem: Suppose (f_n) is a sequence of continuous functions on $[a, b]$ that converges uniformly to a function f on the same interval. Then $\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx$.

(See notes or the text.)

3. The inverse hyperbolic tangent function happens to have series $\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$ (and the series has radius of convergence $R = 1$). Find the power series of the derivative of this function (valid for $|x| < 1$).

This is just

$$\begin{aligned}
\frac{d}{dx} \tanh^{-1} x &= \frac{d}{dx}\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right) \\
&= 1 + x^2 + x^4 + x^6 + \cdots = \sum_{n=0}^{\infty} x^{2n} = \frac{1}{1 - x^2}\,.
\end{aligned}$$

4. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where the radius of convergence of the series is $R > 0$. Let $F(x) = \int_0^x f(t)\, dt$. In the case that $0 < x < R$ prove that $F(x) = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$.

(See notes or the text.)

5. Find the derivative of the function $f(x) = \sqrt{2x+3}$ by using the definition of derivative.

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{\sqrt{2x+3} - \sqrt{2a+3}}{x - a} \\
&= \lim_{x\to a} \frac{\left(\sqrt{2x+3} - \sqrt{2a+3}\right)\left(\sqrt{2x+3} + \sqrt{2a+3}\right)}{(x-a)\left(\sqrt{2x+3} + \sqrt{2a+3}\right)} \\
&= \lim_{x\to a} \frac{(2x+3) - (2a+3)}{(x-a)\left(\sqrt{2x+3} + \sqrt{2a+3}\right)} = \lim_{x\to a} \frac{2(x-a)}{(x-a)\left(\sqrt{2x+3} + \sqrt{2a+3}\right)} \\
&= \lim_{x\to a} \frac{2}{\sqrt{2x+3} + \sqrt{2a+3}} = \frac{2}{2\sqrt{2a+3}} = \frac{1}{\sqrt{2a+3}}.
\end{aligned}$$

6. Prove that if f is differentiable at the number a, then f is continuous at a.

This is $\lim_{x\to a} f(x) = \lim_{x\to a}\left(f(a) + \frac{f(x)-f(a)}{x-a}(x-a)\right) = f(a) + f'(a)(0) = f(a)$.

7. State and prove the Mean Value Theorem for Derivatives.

(See notes or the text.)

8. Find $g'(a)$ where g is the inverse function for $f(x) = \tanh x$. (You are allowed to use the following facts: The hyperbolic tangent function is differentiable and its derivative is $\frac{d}{dx}\tanh x = \text{sech}^2 x$. Also, $\tanh^2 x = \text{sech}^2 x - 1$. Note, the hyperbolic tangent function is increasing on $\mathbb{R}$, so it has a well-defined inverse function.)

Note, there was an error in the statement of the problem: The hyperbolic identity should read $\tanh^2 x = \text{sech}^2 x + 1$. (Of course, this didn't count against anyone's answer.)

The solution is $g'(a) = \frac{1}{f'(g(a))} = \frac{1}{\text{sech}^2\left(\tanh^{-1} a\right)} = \frac{1}{\tanh^2\left(\tanh^{-1} a\right) - 1} = \frac{1}{a^2-1}$. Compare problem 3, by the way.

9. State the Cauchy Mean Value Theorem (or the Extended Mean Value Theorem as given in the text).

(See notes or the text.)

10. Use L'Hospital's Rule to find the following limit: $\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x$.

If $L = \lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x$ then

$$\begin{aligned}
\ln L &= \lim_{x\to\infty} \ln\left[\left(1+\tfrac{4}{x}\right)^x\right] = \lim_{x\to\infty} x\ln\left(1+\tfrac{4}{x}\right) \\
&= \lim_{x\to\infty} \frac{\ln\left(1+\frac{4}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{4}{x}}\left(-\frac{4}{x^2}\right)}{\frac{-1}{x^2}} \\
&= \lim_{x\to\infty} \frac{4}{1+\frac{4}{x}} = 4.
\end{aligned}$$

Therefore, $L = e^4$.